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AIRFOIL CASCADE IN UNSTEADY EDDY FLOW

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1. Introduction. In passing through a row of blades in a turbomachine, a nonuniform fluid stream undergoes a change. To date, the question of how drastic the change is and how it affects the hydrodynamic characteristics of the row has not been adequately studied. Theoretical analyses have been performed mainly with the use of a two-dimensional model of flow under the assumption that the flow is nonuniform and the loading on the airfoils is small. In this statement of the problem, known as the problem of a cascade in an unsteady eddy flow, the disturbance of the flow induced by the airfoils is of purely potential character. Due to the exponential decrease of the disturbance, the flow nonuniformity upstream and downstream the cascade remains invariant. In recent years the availability of powerful computers made it possible to perform the calculations on the basis of the full Euler or Navier–Stokes equations to obtain the flow pattern behind the cascade [1, 2]. In view of the complexity of the model, only sporadic results have been obtained, which does not enable us to study the dependence of the flow structure on the basic parameters of the cascade.

In the present paper, as in the problem of a cascade in unsteady eddy flow, the nonuniformity is assumed to be small, but the restriction on the loading on airfoils is removed. The airfoils may be of arbitrary shape. We consider the problem linearized on a steady stream, which corresponds to constant (at infinity) flow around the cascade.

2. Statement of the Problem. Let us consider an airfoil cascade in the stream of an ideal incompressible liquid in the plane of the complex variable $z = x + iy$. We assume the arbitrary airfoils in the cascade to be smooth or with a sharp trailing edge. Let us suppose that the complex stream velocity at infinity ahead of the cascade can be presented as $\mathbf{V} = \mathbf{V}_{1\infty} + \mathbf{J}$, where $\mathbf{V}_{1\infty} = \text{const}$, and $\mathbf{J} = \mathbf{J}(x, y + ut) = \mathbf{J}(x, y + ut + h_1)$ is the small nonuniformity. In the general case this is eddy nonuniformity ($\operatorname{rot}\vec{J} \neq 0$) propagating in the y direction as a periodic traveling wave with velocity u (Fig. 1). We suggest an arbitrary period of nonuniformity h_1 and a cascade pitch h_2 that satisfy the condition $H = N_1h_1 = N_2h_2$ (H is the full period, N_1 and N_2 are integers). Let us neglect the transient vortex wakes which trail down the airfoils due to the change in circulation (quasisteady-state statement of the problem). Moreover, we assume that there is no reverse stream near the cascade, while the eddying at infinity ahead of the cascade is equal to zero:

$$\lim_{x\to-\infty} \int_y^{y+h_1} \Omega_\infty(x, \eta)V_{1\infty x}d\eta = 0,\ V_{1\infty x} = \lim_{x\to-\infty} V_{0x},\ \Omega_\infty = \frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y}. \tag{2.1}$$

Let us present the complex velocity of a liquid as $\bar{V}(z, t) = \bar{V}_0(z) + \bar{v}(z, t)$, $|\bar{v}| << |\bar{V}_0|$, where $\bar{V}_0$ is the complex velocity of the constant (at infinity) flow past the cascade $\bar{V}_{1\infty}$, and $\bar{v}$ is the sought-for small addend. Then we decompose $\bar{v}$ into the vortex $\bar{v}_1$ and the potential $\bar{v}_2$ components: $\bar{v} = \bar{v}_1 + \bar{v}_2$. The complex velocity $\bar{v}_1$ is induced by the vortex nonuniformity specified at infinity ahead of the cascade $\bar{J}$ and expressed by the Biot–Savart formula:

$$\bar{v}_1(z, t) = \frac{1}{2\pi i}\iint_S \frac{\Omega(\xi, \eta, t)d\xi d\eta}{z - \zeta},\ \zeta = \xi + i\eta,\ \Omega = \frac{\partial v_{1y}}{\partial x} - \frac{\partial v_{1x}}{\partial y} \tag{2.2}$$

where S is the cascade exterior.

From the assumptions on the absence of reverse stream and zeroth eddying at infinity ahead of the cascade, the following estimate is valid [3]:

$$|\bar{v}_1| < 0{,}5H(\max|\Omega_\infty V_{1\infty x}|/u + 0{,}5\max|\Omega_\infty|), \tag{2.3}$$

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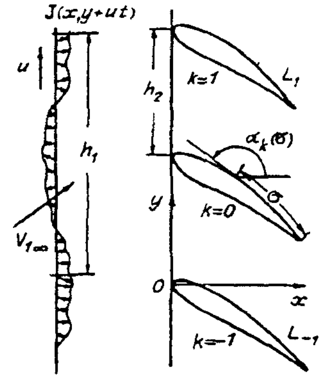

Fig. 1

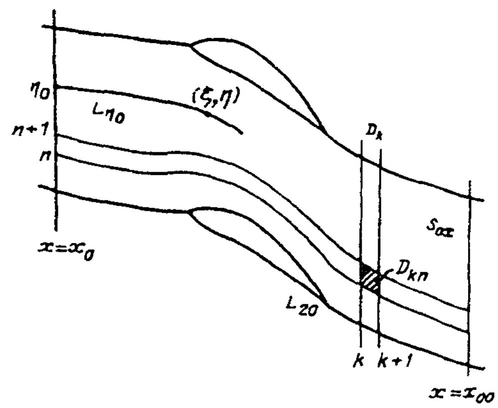

Fig. 2

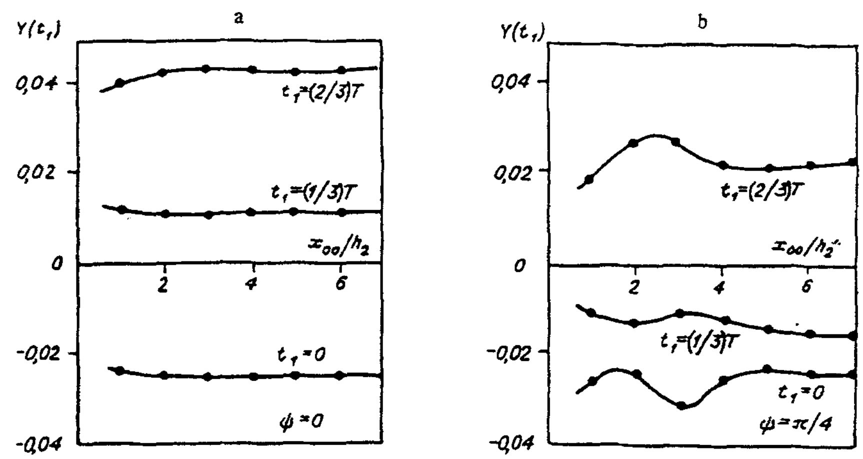

Fig. 3

This estimate ensures the smallness of the nonuniformity $\overline{v}_1$ in the whole stream domain on the basis of the data on its smallness at infinity ahead of the cascade. Inequality (2.3) allows for the use of the linearized Helmholtz equation

$$\frac{\partial\Omega}{\partial t} + \frac{\partial\Omega}{\partial x}V_{0x} + \frac{\partial\Omega}{\partial y}V_{0y},\ \overline{V}_0 = V_{0x} - iV_{0y}, \tag{2.4}$$

TABLE 1

λ_γ	γ_{11}	γ_{12}	γ_{21}	γ_{22}	γ_{31}	γ_{32}	γ_{41}	γ_{42}
Not accounting for evolving wakes								
4,16	−0,0083	−0,0036	0,0614	−0,0459	−0,0094	−0,0617	0,0037	0,0074
Accounting for evolving wakes								
4,11	−0,0076	−0,004	0,0589	−0,0428	−0,0117	−0,0708	0,0081	0,0059

TABLE 2

λ_γ	γ_{11}	γ_{12}	γ_{21}	γ_{22}	γ_{31}	γ_{32}	γ_{41}	γ_{42}
Not accounting for evolving wakes								
0,238	−0,0511	−0,178	−0,0093	−0,0674	0,099	−0,141	0,003	−0,1159
Accounting for evolving wakes								
0,283	−0,0827	−0,129	0,014	0,0076	0,262	0,136	−0,011	0,0068

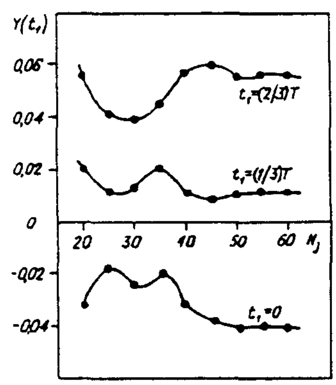

Fig. 4

which, given the known vorticity Ω_∞ at infinity ahead of the cascade, enables one to determine the vorticity in the whole stream domain S.

The complex velocity $\bar{v}_2(z, t)$, by definition, is an analytic function with respect to z in the stream domain S, which satisfies the following conditions: the liquid does not flow past the airfoils $v_{2n}(z, t) = -v_{1n}(z, t)$ [$z \in L_k$, L_k is the contour of the k-airfoil in the cascade, $k = 0, \pm 1, \ldots$, n is the normal to it (which follows from $\bar{V}_n(z, t) = V_{0n}(z, t) = 0$)], the stream is periodic in the y direction $\bar{v}_2(z, t) = \bar{v}_2(z + iH, t)$, the velocity of the liquid at infinity and at fixed points of trailing edges of smooth airfoils is equal to zero, or the velocity is finite at sharp trailing edges of the airfoils (Zhukovskii's condition).

From the Euler equation written in Lamb's form we obtain the formula for calculating the pressure at the airfoil cascade:

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{V}^2\right) - \mathbf{V} \times \operatorname{rot}\mathbf{V} = -\frac{1}{\rho}\nabla p. \tag{2.5}$$

Integrating (2.5) along the airfoil and taking into account the condition $V_n = 0$, we obtain

$$p(\sigma, t) = -\frac{1}{2}\rho V^2(\sigma, t) + p(0, t) - \rho\frac{\partial}{\partial t}\int_0 V(\sigma, t)dt, \tag{2.6}$$

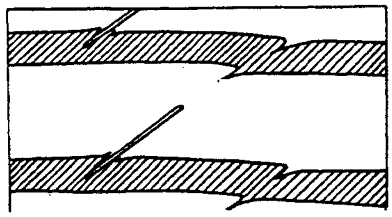

Fig. 5

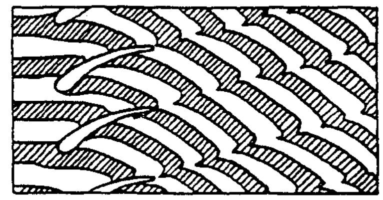

Fig. 6

where σ is the arc coordinate of the initial airfoil measured from the trailing edge $V(0, t) = 0$. The expressions for the total hydrodynamic force and moment are obtained from (2.6) by integrating the pressure along the contour of the airfoil.

3. Numerical Method. To solve the problem (2.2) and (2.4) we expand the vorticity Ω and velocity $\bar{v}_1$ in Fourier series with respect to time ($i \neq j$):

$$\Omega(z, t) = \sum_{r=0}^{\infty} \Omega_r(z)\exp(-jr\omega_1 t), \qquad \bar{v}_1(z, t) = \sum_{r=0}^{\infty} \bar{v}_r(z)\exp(-jr\omega_1 t), \ \omega_1 = 2\pi u/h_1. \tag{3.1}$$

Substituting (3.1) into (2.1) with the use of the relationship

$$\Omega(z + imh_2, t) = \Omega(z, t + mh_2/u)$$

and the equality

$$\sum_{k=-\infty}^{\infty} \frac{1}{z - \zeta - ikH} = \frac{\pi}{H}\operatorname{cth}\frac{\pi}{H}(z - \zeta),$$

we obtain

$$\bar{v}_r(z) = \frac{1}{2Hi}\int\int_{S_0} \Omega_r(\xi) \sum_{m=0}^{N_2-1} \exp(-jrm\psi)\operatorname{cth}\frac{\pi}{H}(z - \zeta - imh_2)d\xi d\eta. \tag{3.2}$$

Here $\psi = 2\pi N_1/N_2$, S_0 is the interblade domain of the initial airfoil cascade (Fig. 2). Due to the exponential decrease of the functions $(\bar{V}_0 - \bar{V}_{1\infty})$ and $\bar{v}_2$ as the distance x increases, one may approximately assume, in calculating the integral (3.2), that already at some finite distance $|x_0|$ upstream of the cascade front $\bar{v}_1(x, t) \approx \bar{J}(x, y + ut)$ and $\Omega(z, t) \approx \Omega_{\infty}(z, t)$, with $x < x_0$. Along with the condition (2.1), this makes it possible to replace the integration with respect to the infinite domain S_0 by that with respect to the semi-infinite domain $S_x = \dot{S}_0 \cup \{x > x_0\}$.

Let us then denote the ordinate of the point of intersection of the axis $x = x_0$ with the streamline with velocity $\bar{V}_0$ which passes through the point (ξ, η) as $\eta_0 = \eta_0(\xi, \eta)$, and the time it takes for the elements of volume to travel from the point (x_0, η_0) to the point (ξ, η) moving with the velocity $\bar{V}_0$, as $t_0 = t_0(\xi, \eta)$:

$$t_0(\xi, \zeta) = \int_{L_{\eta_0\xi\eta}} \frac{ds}{\underline{V_0(s)}} \tag{3.3}$$

($L_{\eta_0\xi\eta}$ is the segment of the streamline up to the point (ξ, η)). Then from Eq. (2.4) $\Omega(\xi, \eta, t) = \Omega(x_0, \eta_0, t - t_0)$, and we obtain the formula

$$v_r(z) = \frac{\Omega_{r0}}{2Hi}\int_{S_{x0}} \exp(jr\omega_1(t_0 - \eta_0/u))\sum_{m=0}^{N_2-1} \exp(-jrm\psi) \times \operatorname{cth}\frac{\pi}{H}(z - \zeta - imh_2)d\xi d\eta \tag{3.4}$$

where Ω_{r0} is the coefficient of the Fourier expansion

$$\Omega(x_0, y, t) = \sum_{r=0}^{\infty} \Omega_{r0}\exp(-jr\omega_1(t - y/u)). \tag{3.5}$$

Formulas (3.3)-(3.5) allow for the eddy-velocity component $\bar{v}_1$ to be calculated from the known main stream $\bar{V}_0$ and the nonuniformity $\bar{J}$ of the windstream. Given the function $\bar{v}_1$, the problem of searching for potential velocity component $\bar{v}_2$ can be solved as in [4].

4. Convergence of the Algorithm, Calculation Examples. When computing the integral (3.4), the semi-infinite domain S_{x0} was truncated to the right, i.e., the computations were performed up to $x = x_{00}$ (Fig. 2). To check the convergence of this parameter of the algorithm, the computations were carried out for $x_{00}/h_2 = 1, 2, 3, \ldots, 10$. The values of the circumferential force component $Y(t_1)$ for three fixed instants of time $t_1 = 0, T/3, 2T/3$ ($T = h_1/u$ is the time period) are presented in Fig. 3. With $\psi = 2\pi$ (one airfoil in a period), the forces and moments showed agreement up to the third decimal place in all calculations already when $x_{00}/h_2 > 3$ (Fig. 3a). With $\psi = \pi/4$ (four airfoils in a period), to ensure accuracy, the calculations should be performed up to $x_{00}/h_2 = 5$-6 (Fig. 3b). This result is consistent with known theoretical and experimental data which indicate that, as the distance x increases, the influence of the flow nonuniformity is greater the smaller the phase shift ψ between the adjacent airfoils.

The second parameter governing the calculation accuracy is the number of cells in the calculational domain S_{x0} (mesh density). In the algorithm the mesh density was prescribed by the number N of points of splitting of the entry $x = x_0$. The values of the stream functions ψ_{0n} and the steady flow $\bar{V}_0$ were calculated at these points (x_0, η_{0n}), $n = 1, \ldots, N$. The unit calculation cells D_{kn} were obtained from the intersection of the domain between two neighboring streamlines ψ_{on} and ψ_{0n+1} with the vertical section D_k. The cells D_{kn} were constructed smaller the closer they were to the cascade airfoils (see Fig. 2).

Values of the force $Y(t_1)$ ($t_1 = 0, T/3, 2T/3$) for $N = 20, 25, 30, \ldots, 60$ are presented in Fig. 4. The calculations point to considerably slow convergence of the algorithm in the mesh density. Use of an IBM AT-286 computer made difficult the calculations for $N > 60$ (for $N = 60$ the total number of cells is 10,000). For comparison, the number of cells used in calculating the two-dimensional flows in the cascade domain in [1, 2] is greater by one or two orders of magnitude. The satisfactory calculation accuracy for ~10,000 cells is due to the linear character of the problem. The algorithm takes into account the fact that the value of vorticity is preserved along the stream tube of steady flow. Since the stream tubes can be calculated with sufficient accuracy in advance, the one-dimensional problem is solved for each of them. The number of stream tubes is N.

Figure 5 presents as a test case the results of calculation of the limiting case for an airfoil cascade flown at small angle of attack (at the instant of time t = 0). As one would expect, in this case the evolution of vortex wakes, which is specified by the function J, appeared to be small. The results of calculation of transient forces are close to those for the model neglecting the evolution [4]. Table 1 shows the results of comparison of separate harmonics (first four) Y_{n1} and Y_{n2} of the circumferential force component

$$Y(t) = Y_0 + \sum_{n=1}^{\infty} (Y_{n1}\cos\omega t + Y_{n2}\sin\omega t),$$
$$\omega = 2\pi/h,$$

as well as the value $\lambda_y = [(\max(Y(t)) - \min(Y(t))]/Y_0$ which characterizes the total level of the exciting forces. The slight discrepancy in the data is due to the fact that the program was designed for calculating arbitrary airfoils. Therefore, the plate had to be simulated also by the airfoil, which results in a slight deformation of the vortex wakes.

The evolution of the wakes when they pass through a typical compressor cascade is shown in Fig. 6. A comparison with the results of [5] is presented in Table 2. When passing through the cascade the vortex wakes are subject to significant changes, which result (Table 2) in a significant change in the values of amplitude of separate harmonics of the exciting forces. Along with this the total level of the exciting forces λ_y varies slightly. A qualitatively similar result was obtained when the compressor cascade was replaced by a cascade of circles (heavily loaded cascade). In this case the evolution of wakes appeared to be even more pronounced and the relative contribution of separate harmonics of the excitation force was greater. However, the parameter λ_y changed slightly as before. This is presumably due to the fact that the evolution of wakes leads to a substantial redistribution of separate harmonics in the wake itself while retaining the integral characteristic of the velocity dip in it as a whole.

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